THERMOCAPILLARY CONVECTION IN A LAYER OF LIQUID

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Convection in a slag at the surface of liquid metal and convection in a liquid metal without slag are described respectively by a model of convection in a liquid layer with a free surface and by a model of convection at the surface of a solid. The conditions for convection in a liquid and at the surface of a solid produced by the thermocapillary effect have been considered earlier in [1], and the combined gravitational and thermocapillary effect has been considered in [2].

The liquid layer is assumed sufficiently thin to make the Archimedes force in the equations of motion negligible but sufficiently thick to allow thermocapillary convection to occur at moderate temperature gradients with the viscosity assumed not to vary with temperature. The velocity field in a convection cell of slag or metal is calculated here and the results are then applied to the determination of diffusive flow across the surface of the liquid.

1. Fundamental Equations. We consider a liquid layer confined between the planes $z = 0$ (upper surface) and $z = h$ (lower surface). The liquid in the layer is assumed at rest. Under the condition of a uniform temperature over the boundary surfaces the temperature profile across the liquid will be linear; $T = T_0 + Az$.

We introduce the following dimensionless variables:

$$x = h\xi, \quad y = h\eta, \quad z = h\zeta, \quad t' = (h^2/\nu)\,t$$
$$v_x = (\nu/h)\,u, \quad v_y = (\nu/h)\,v, \quad v_z = (\nu/h)\,w$$
$$p = \rho\,(\nu/h)^2\,\Pi, \qquad T = T_0 + Ah\,(\zeta + \theta)\,.$$

Here ν is the kinematic viscosity of the liquid, ρ is its density, p is the pressure, and t' is the time.

The equations of motion for an incompressible viscous liquid and the equation of convective heat transfer, both in dimensionless form, are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} = -\nabla\Pi + \Delta\mathbf{v}, \qquad \operatorname{div}\mathbf{v} = 0$$
$$\frac{\partial\theta}{\partial t} + \mathbf{v}\nabla\theta = \frac{1}{P}\,\Delta\theta - w \qquad \left(P = \frac{\nu}{\chi}\right). \tag{1.1}$$

Here χ is the thermal diffusivity. Thermocapillary convection occurs as a result of the fact that the coefficient of surface tension varies with temperature and, therefore, occurs when the temperature at the boundary surfaces is not distributed uniformly.

When the temperature gradient exceeds a certain critical value, then, as will be shown subsequently, a uniform temperature distribution is unstable. The motion of the liquid, which begins at small deviations from a uniform temperature distribution, causes this deviation to increase and a nonuniform temperature distribution to be established over the surfaces. Moreover, the entire liquid layer will break down into cells, just as in the case of gravitational convection [3]. The liquid rises along the axis of each cell and drops along the edges, or vice-versa.

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It is well known [4, 5] that a change in the surface tension σ along a boundary surface produces tangential forces whose magnitude per unit surface area is $\nabla\sigma$. Assuming that the temperature variation along the z = 0 surface is small, we may let $\nabla\sigma = -\gamma\nabla T$ (γ = const).

The condition of impermeability for the surface and the condition of continuity for the tangential components of the stress tensor signify that at z = 0

$$v_z = 0, \quad \mu\frac{\partial v_x}{\partial z} = \gamma\frac{\partial T}{\partial x}, \quad \mu\frac{\partial v_y}{\partial z} = \gamma\frac{\partial T}{\partial y} \quad (\mu = \rho\nu). \tag{1.2}$$

From the continuity equations and (1.2) follows

$$\gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T + \mu\frac{\partial^2 v_z}{\partial z^2} = 0$$

or, in dimensionless form with $\zeta = 0$,

$$w = 0, \qquad \frac{\partial^2 w}{\partial \zeta^2} = -C\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\theta \qquad \left(C = \frac{\gamma A h^2}{\mu\nu}\right). \tag{1.3}$$

A heat transfer is assumed to occur at the upper surface of the liquid with the surrounding medium which yields a corresponding boundary condition:

$$\varkappa\,\partial T/\partial z = Q_0 + q\,(T - T_0).$$

Here $\varkappa$ is the thermal conductivity of the liquid, Q_0 is the thermal flux density through a unit surface area of the liquid at the surface temperature T_0, and $q(T-T_0)$ is the change in thermal flux density due to a small deviation of the temperature from T_0.

Changing into the dimensionless form yields at $\zeta = 0$ the condition

$$\partial\theta/\partial\zeta = B\theta \qquad (B = qh/\varkappa). \tag{1.4}$$

In analyzing the motion in a slag at the surface of liquid metal, we assume that the metal below has a negligibly low viscosity but a much higher thermal conductivity than the liquid slag. The slag-metal boundary may then be considered free, and its temperature constant:

$$w = \partial^2 w/\partial\zeta^2 = \theta = 0 \qquad \text{at} \quad \zeta = 1, \tag{1.5}$$

For the layer of liquid metal we write analogously

$$w = \partial w/\partial\zeta = \partial\theta/\partial\zeta = 0 \qquad \text{at} \quad \zeta = 1 \tag{1.6}$$

assuming here that the conditions of adhesion prevail at the solid surface and that, when the liquid metal has a high thermal conductivity, the thermal flux does not depend on changes in the surface temperature.

2. The Linearized Steady-State System of Equations. Since the system of equations and boundary conditions is homogeneous, the velocity and temperature field will be determined exactly except for an arbitrary factor henceforth called the amplitude.

An analytical solution of the linearized steady-state system will allow us to determine C_* – the critical value of parameter C – at which a steady-state solution is possible.

An analysis of the standstill stability under small perturbations is, by the substitutions

$$\mathbf{v} = \mathbf{v}_1 \exp(-st), \quad \Pi = \Pi_1 \exp(-st), \quad \theta = \theta_1 \exp(-st)$$

($\mathbf{v}_1$, Π_1, and θ_1 are independent of time), reduced to the problem of finding the eigenvalues for the system of equations

$$-s\mathbf{v}_1 = -\nabla\Pi_1 + \Delta\mathbf{v}_1, \quad \operatorname{div}\mathbf{v}_1 = 0$$
$$-s\theta_1 = P^{-1}\Delta\theta_1 - w_1$$

with the appropriate boundary conditions.

The standstill is not stable if among the eigenvalues is at least one with a nonpositive real component Re $s \le 0$. Therefore, the beginning of stability is associated with the appearance of solutions where Re $s = 0$.

The study is concerned with steady-state convection resulting from instability, and therefore, we must seek the solution with $s = 0$. The corresponding values of the temperature gradient A and of the parameter C will be the critical ones A_* and C_*. Steady-state convection is not possible when $C < C_*$.

Until now apparently, the stability and the beginning of thermocapillary convection have not been studied thoroughly enough. The sufficiently general and convincing ideas stated by L. D. Landau in his study of turbulence [5] can be useful, however, in hypothesizing on the character of this kind of motion.

Letting C be slightly above critical, one may assume, according to L. D. Landau, that the amplitudes of the velocity field are proportional to the quantity $\varepsilon = \sqrt{(C - C_*/C_*}$ in the linearized as well as in the nonlinear steady-state problem. The amplitude of the initial convection current is small, and a solution of the linearized system of equations will yield the solution to the nonlinear system by the method of small-parameter perturbation:

$$\begin{aligned} \mathbf{v} &= \varepsilon \mathbf{v}^{(0)} + \varepsilon^2 \mathbf{v}^{(1)} + \varepsilon^3 \mathbf{v}^{(2)} + \ldots \\ \Pi &= \varepsilon \Pi^{(0)} + \varepsilon^2 \Pi^{(1)} + \varepsilon^3 \Pi^{(2)} + \ldots \\ \theta &= \varepsilon \theta^{(0)} + \varepsilon^2 \theta^{(1)} + \varepsilon^3 \theta^{(2)} + \ldots . \end{aligned} \tag{2.1}$$

Here $\varepsilon \mathbf{v}^{(0)}$, $\varepsilon \Pi^{(0)}$, $\varepsilon \theta^{(0)}$ are the solution to the linearized problem.

The linearized steady-state system of equations

$$\Delta \mathbf{v} - \nabla \Pi = 0, \qquad \operatorname{div} \mathbf{v} = 0, \qquad \Delta \theta - Pw = 0 \tag{2.2}$$

can be reduced to a system of equations in terms of functions of w and θ:

$$\Delta \Delta w = 0, \qquad \Delta \theta = Pw \tag{2.3}$$

by eliminating u, v, and Π with the aid of the second equation in (2.2).

According to [3], a liquid layer during steady-state convection may be assumed to break down into cylindrical cells with axial symmetry. In cylindrical coordinates, w and θ will be functions of two variables ρ and ζ, where ρ is the dimensionless radial coordinate (ratio of the radial dimension to the layer thickness h).

System (2.3) with the corresponding boundary conditions is solved by the method of separating the variables:

$$w = kJ_0(a\rho)\, W(\zeta), \qquad \theta = kPJ_0(a\rho)\, \Theta(\zeta). \tag{2.4}$$

Here k is the amplitude of the velocity field, J_0 is the zero-order Bessel function, and a is the separation-of-variables constant.

Functions W and Θ satisfy the equations

$$\left(\frac{d^2}{d\zeta^2} - a^2\right)^2 W = 0, \qquad \left(\frac{d^2}{d\zeta^2} - a^2\right) \Theta = W. \tag{2.5}$$

The condition that the velocity field in a convection cell is bounded and that the radial velocity component v_ρ vanishes at the lateral boundary of a cylindrical cell will determine the appropriate zero-order Bessel function among the set of solutions to the Bessel equations obtained for the radial variable after separation, and it requires that a^2 be real.

Assuming that the radial component of the velocity field first becomes zero at $\rho \ne 0$ at the cell boundary, we see that obviously the dimensionless radius of the cell is equal to $\rho_0 = x_{11}/a_*$, where x_{11} is the small-

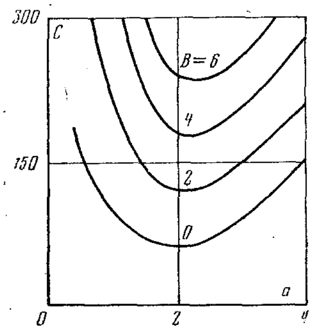

Fig. 1

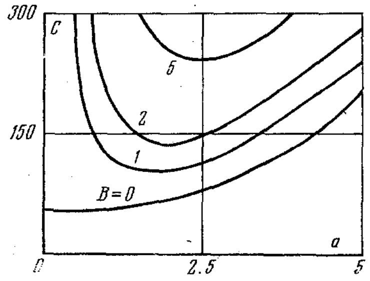

Fig. 2

est positive root of the first-order Bessel function $J_1(x)$ and a_* is the value of parameter a which corresponds to the beginning of convection, i.e., to C_*.

It is now evident that the value of a in Eqs. (2.4) is not arbitrary but such as to make $a\rho_0 = x_{1l}$, where x_{1l} is the l-th positive root of the first-order Bessel function.

In this study we analyze convection at values of C only slightly above critical, and therefore, a is assumed equal to a_*.

Equations (2.5) can be solved for two sets of boundary conditions corresponding to two cases considered here: convection in a slag and convection in a metal.

At the upper boundary $\zeta = 0$ the conditions are

$$\frac{d^2}{d\zeta^2} W = a^2 C\Theta, \quad \frac{d}{d\zeta}\Theta = B\Theta, \quad W = 0. \tag{2.6}$$

These conditions are supplemented by additional ones:

$$W = \frac{d^2}{d\zeta^2} W = \Theta = 0 \quad \text{at} \quad \zeta = 1 \tag{2.7}$$

for convection in a slag and

$$W = \frac{d}{d\zeta} W = \frac{d}{d\zeta}\Theta = 0 \quad \text{at} \quad \zeta = 1 \tag{2.8}$$

for convection in a metal.

The solution to Eqs. (2.5) with conditions (2.6) and (2.7) for convection in a slag is

$$W = \operatorname{sh} a\zeta + \zeta \operatorname{sh}^2 a \operatorname{sh} a\zeta - \zeta \operatorname{sh} a \operatorname{ch} a \operatorname{ch} a\zeta \tag{2.9}$$

$$\Theta = \frac{2\operatorname{sh}^2 a}{aC}\left(\operatorname{ch} a\zeta + \frac{B}{a}\operatorname{sh} a\zeta\right) + \frac{1}{a}\int_0^\zeta W(\zeta')\operatorname{sh} a(\zeta - \zeta')\,d\zeta'$$

$$\left(C = \frac{8a\operatorname{sh}^2 a\,(a\operatorname{ch} a + B\operatorname{sh} a)}{(a + \operatorname{sh} a\operatorname{ch} a)\operatorname{sh} a - 2a^2\operatorname{ch} a}\right)$$

$$C = 8a^2\left(1 + \frac{B}{a}\right) \text{ for } a \gg 1, \; C = \frac{45}{a^2}(1 + B) \text{ for } a \ll 1. \tag{2.10}$$

The calculated curves representing C (a, B) are shown in Fig. 1 for various values of B.

The results here indicate that thermocapillary convection in a layer with a free lower surface occurs at critical values C_* and a_* which depend on parameter B. At values $C < C_*$ there occurs no steady-state convection.

If the liquid borders on a solid surface underneath, then the solution to Eqs. (2.5) with conditions (2.6) and (2.8) is

$$W = a\,\text{sh}\,a\zeta + (\text{sh}\,a\,\text{ch}\,a - a)\zeta\,\text{sh}\,a\zeta - \zeta\,\text{sh}^2 a\,\text{ch}\,a\zeta \qquad (2.11)$$

$$\Theta = \frac{\text{sh}\,2a - 2a}{aC}\left(\text{ch}\,a\zeta + \frac{B}{a}\,\text{sh}\,a\zeta\right) + \frac{1}{a}\int_0^\zeta W(\zeta')\,\text{sh}\,a(\zeta - \zeta')\,d\zeta'$$

$$C = \frac{4a\,(a\,\text{sh}\,a + B\,\text{ch}\,a)\,(\text{sh}\,2a - 2a)}{\text{sh}^2 a\,\text{ch}\,a - 2a\,\text{sh}\,a + a^2\,\text{ch}\,a - a^3\,\text{sh}\,a} \qquad (2.12)$$

$$C = 8a^2\,(1 + B\,/\,a) \quad \text{for} \quad a \gg 1$$

$$C = 48\,(1 + B\,/\,a^2) \quad \text{for} \quad a \ll 1 \; .$$

A comparison between Fig. 1 and Fig. 2 (taken from [1]) indicates that convection begins in the first case at values of C approximately 1.5 times lower than in the second case.

3. Amplitude of the Velocity Field. The velocity field for the linearized system of equations can be found exactly except for the amplitude, the determination of which is tied to the existence condition for a solution to the nonlinear system of steady-state equations. Gravitational convection has been analyzed earlier in an analogous manner [6].

If we eliminate C from the boundary conditions, which can be done by replacing θ with a new function $\tau = C\theta$, then the equations of motion and of heat transfer can be written in the operator form as

$$L'X' = F \qquad (3.1)$$

$$L' = \begin{pmatrix} \Delta & 0 & 0 & -\frac{\partial}{\partial\xi} & 0 \\ 0 & \Delta & 0 & -\frac{\partial}{\partial\eta} & 0 \\ 0 & 0 & \Delta & -\frac{\partial}{\partial\zeta} & 0 \\ -\frac{\partial}{\partial\xi} & -\frac{\partial}{\partial\eta} & -\frac{\partial}{\partial\zeta} & 0 & 0 \\ 0 & 0 & -1 & 0 & \frac{1}{PC}\Delta \end{pmatrix}, \quad X = \begin{pmatrix} u \\ v \\ w \\ \Pi \\ \tau \end{pmatrix}, \quad F = \begin{pmatrix} (\mathbf{v}\nabla)\,u \\ (\mathbf{v}\nabla)\,v \\ (\mathbf{v}\nabla)\,w \\ 0 \\ \frac{\mathbf{v}\nabla\tau}{C} \end{pmatrix}$$

Evidently, unlike in the case of gravitational convection [6], operator L' is not a self-adjoint one. If we consider functions $\mathbf{v}$, $\Pi - \zeta\tau$, and τ instead of $\mathbf{v}$, Π, and τ, however, then Eqs. (3.1) will yield a system which can be expressed in operator form with a self-adjoint operator:

$$LX = F \qquad (3.2)$$

$$L = \begin{pmatrix} \Delta & 0 & 0 & -\frac{\partial}{\partial\xi} & -\zeta\frac{\partial}{\partial\xi} \\ 0 & \Delta & 0 & -\frac{\partial}{\partial\eta} & -\zeta\frac{\partial}{\partial\eta} \\ 0 & 0 & \Delta & -\frac{\partial}{\partial\zeta} & -\zeta\frac{\partial}{\partial\zeta} - 1 \\ -\frac{\partial}{\partial\xi} & -\frac{\partial}{\partial\eta} & -\frac{\partial}{\partial\zeta} & 0 & 0 \\ -\zeta\frac{\partial}{\partial\xi} & -\zeta\frac{\partial}{\partial\eta} & -\zeta\frac{\partial}{\partial\zeta} - 1 & 0 & \frac{1}{PC}\Delta \end{pmatrix}$$

$$X = \begin{pmatrix} u \\ v \\ w \\ \Pi - \zeta\tau \\ \tau \end{pmatrix}$$

The system of Eqs. (3.2) can be solved by the method of series expansion in a small parameter ε, where necessarily $1/C = (1 - \varepsilon^2)/C_*$.

In this way we obtain the following system of equations:

$$L_0 X^{(0)} = 0,$$

$$L_0 X^{(1)} = F^{(1)}, \qquad L_0 X^{(2)} = F^{(2)} . \qquad (3.3)$$

In accordance with the Fredholm alternative, nonhomogeneous systems of equations like (3.3) have solutions $X^{(1)}$, $X^{(2)}$ only if the right-hand sides of these equations $F^{(1)}$ and $F^{(2)}$ are orthogonal to the $X^{(0)}$ solution to the homogeneous system of equations

$$\int\left[\mathbf{v}^{(0)}(\mathbf{v}^{(0)}\nabla)\mathbf{v}^{(0)}+\frac{1}{C_*}\tau^{(0)}\mathbf{v}^{(0)}\nabla\tau^{(0)}\right]d^3r=0$$

$$\int\left[\mathbf{v}^{(0)}(\mathbf{v}^{(0)}\nabla)\mathbf{v}^{(1)}+\mathbf{v}^{(0)}(\mathbf{v}^{(1)}\nabla)\mathbf{v}^{(0)}+\frac{1}{C_*}\tau^{(0)}(\mathbf{v}^{(0)}\nabla\tau^{(1)}+\mathbf{v}^{(1)}\nabla\tau^{(0)})+\tau^{(0)}w^{(0)}\right]d^3r=0 \tag{3.4}$$

$$(d^3r=2\pi\rho d\rho d\zeta).$$

The absence of a normal velocity component at a cell surface and the continuity equation allow a transformation of (3.4) into

$$\int\left[\mathbf{v}^{(0)}(\mathbf{v}^{(0)}\nabla)\mathbf{v}^{(1)}+\frac{1}{C_*}\tau^{(0)}\mathbf{v}^{(0)}\nabla\tau^{(1)}+\tau^{(0)}w^{(0)}\right]d^3r=0. \tag{3.5}$$

Condition (3.5) is satisfied only when the amplitude of the velocity field in a convection cell of a liquid has a definite value.

Indeed, the $X^{(1)}$ solution to the nonlinear system of Eqs. (3.3) will be sought in the form

$$\mathbf{v}^{(1)}=\sum_{n,l}\alpha_{nl}\mathbf{v}_{nl},\quad \Pi^{(1)}=\sum_{n,l}\beta_{nl}\Pi_{nl},\quad \tau^{(1)}=\sum_{n,l}\gamma_{nl}\tau_{nl}. \tag{3.6}$$

Here $\mathbf{v}_{nl}$ is a vector with components $v_{nl\rho}$, 0, w_{nl} in cylindrical coordinates:

$$v_{nl\rho}=-(\pi n/a_l)J_1(a_l\rho)\cos\pi n\zeta,\quad w_{nl}=J_0(a_l\rho)\sin\pi n\zeta \tag{3.7}$$

$$\tau_{nl}=-\frac{(a_l^2+\pi^2n^2)^2}{a_l^2}J_0(a_l\rho)\sin\pi n\zeta,\quad \Pi_{nl}=-\pi n\frac{a_l^2+\pi^2n^2}{a_l^2}J_0(a_l\rho)\cos\pi n\zeta$$

$$a_l=x_{1l}/\rho_0\qquad (l,n=1,2,\ldots)$$

These functions satisfy the equations

$$\Delta\mathbf{v}_{nl}=\nabla\Pi_{nl}+\tau_{nl}\mathbf{e}_\zeta,\qquad \operatorname{div}\mathbf{v}_{nl}=0$$

$$\Delta\tau_{nl}=R_{nl}w_{nl}\qquad (R_{nl}=(a_l^2+\pi^2n^2)^3/a_l^2)$$

Here $\mathbf{e}_\zeta$ is the locus in the direction of the ζ axis. With system (3.7) being complete in the $0\le\rho\le\rho_0$, $0\le\zeta\le1$ range and with the aid of the continuity equation we can seek $X^{(1)}$ in the form (3.6).

Insertion of (3.6) into system (3.3) and a scalar multiplication of this system by the vector whose components are $\mathbf{v}_{mk}$, Π_{mk}, τ_{mk} will yield the following equations:

$$\sum_{n,l}\alpha_{nl}W_{mknl}=V_{mk},\qquad \sum_{n,l}\left(-\alpha_{nl}+\frac{1}{PC_*}R_{nl}\gamma_{nl}\right)W_{nlmk}=T_{mk} \tag{3.8}$$

$$W_{mknl}=\int w_{mk}\tau_{nl}d^3r$$

$$V_{mk}=\int\mathbf{v}_{mk}(\mathbf{v}^{(0)}\nabla)\mathbf{v}^{(0)}d^3r,\qquad T_{mk}=\frac{1}{C_*}\int\tau_{mk}\mathbf{v}^{(0)}\nabla\tau^{(0)}d^3r$$

With the aid of (3.7) we can obtain

$$W_{mknl}=W_{nl}\delta_{mn}\delta_{kl}\qquad \left(\delta_{mn}=\begin{cases}1, & m=n\\ 0, & m\ne n\end{cases}\right)$$

$$(W_{nl}=-{}^1\!/_2\pi\,(a_l+\pi^2n^2/a_l)^2\rho_0^2J_0^2(x_{1l}))$$

From (3.8) we have

$$\alpha_{nl}=V_{nl}/W_{nl},\qquad \gamma_{nl}=PC_*(T_{nl}+V_{nl})/R_{nl}W_{nl}. \tag{3.9}$$

Inserting expressions (2.4) and (3.7) into (3.9) will yield the following result:

$$\alpha_{nl} = \lambda_{nl} a_l \left[(2a_*^2 + \pi^2 n^2 A_{1l}) B_{1n} + \pi n (1 - 2A_{1l}) B_{2n} \right] \tag{3.10}$$

$$\gamma_{nl} = \lambda_{nl} \varphi_{nl} P C_* \left[a_l^3 (2a_*^2 + \pi^2 n^2 A_{1l}) \varphi_{nl}^2 B_{1n} + \pi n a_l^3 (1 - 2A_{1l}) \varphi_{nl}^2 B_{2n} + \right.$$
$$\left. + a_* a_l^2 (A_{2l} - A_{3l}) B_{3n} + \pi n a_* a_l^2 A_{2l} B_{4n} \right]$$

$$\lambda_{nl} = -\frac{4k^2 \varphi_{nl}^2 L_l}{\varepsilon^2 x_{11}^2 J_0^2 (x_{1l})}, \quad \varphi_{nl} = \frac{1}{a_l^2 + \pi^2 n^2}, \quad L_l = \int_0^{\rho_0} J_1^2 (a_* \rho) J_1 (a_l \rho) d\rho$$

$$A_{1l} = \frac{2}{4 - \delta_l^2}, \quad A_{2l} = \frac{4}{\delta_l (4 - \delta_l^2)}, \quad A_{3l} = \frac{4 - 2\delta_l^2}{\delta_l (4 - \delta_l^2)}, \quad \delta_l = \frac{x_{1l}}{x_{11}}$$

$$B_{1n} = \int_0^1 W \frac{dW}{d\zeta} \sin \pi n \zeta d\zeta, \quad B_{2n} = \int_0^1 \left(\frac{dW}{d\zeta} \right)^2 \cos \pi n \zeta d\zeta$$

$$B_{3n} = \int_0^1 \frac{dW}{d\zeta} \Theta \sin \pi n \zeta d\zeta, \quad B_{4n} = \int_0^1 W \Theta \cos \pi n \zeta d\zeta$$

If series (3.6) is now inserted into the condition of solvability (3.5), then

$$\sum_{n,\, l} \int \left[\alpha_{nl} \mathbf{v}^{(0)} (\mathbf{v}^{(0)} \nabla) \mathbf{v}_{nl} + \frac{\gamma_{nl}}{C_*} \tau^{(0)} \mathbf{v}^{(0)} \nabla \tau_{nl} \right] d^3 r + \int \tau^{(0)} w^{(0)} d^3 r = 0 . \tag{3.11}$$

It is assumed that most significant in series (3.6) are the terms with the first nonzero values of coefficients α_{11}, γ_{11}.

The integrals in (3.11) may be expressed as

$$\int \mathbf{v}^{(0)} (\mathbf{v}^{(0)} \nabla) \mathbf{v}_{11} d^3 r = \frac{2\pi k^2 L_1}{\varepsilon^2 a_*^3} \{ B_{11} [(a_*^2 - \pi^2) A_{11} - 2a_*^2 A_{21}] + \pi B_{21} (1 - A_{31}) \} \tag{3.12}$$

$$\int \tau^{(0)} \mathbf{v}^{(0)} \nabla \tau_{11} d^3 r = -\frac{2\pi k^2 L_1}{\varepsilon^2 a_*^3 \varphi_{11}^2} C_* (A_{11} B_{31} + \pi \varphi_{11} A_{21} B_{41})$$

$$\int \tau^{(0)} w^{(0)} d^3 r = \frac{k^2}{\varepsilon^2} C_* \pi \rho_0^2 J_0^2 (x_{11}) B_{40} \quad \left(L_1 = \frac{0.31}{a_*} \right)$$

The integrals in (3.10) and (3.12) with respect to the variable ζ were calculated approximately, using the first terms of the Fourier series into which the integrand functions had been expanded and extracting the linear components for faster convergence.

The amplitude of the convection velocity field in a slag ($P \gg 1$) calculates as follows

$$k = \pm \frac{2.3\varepsilon}{P} \frac{a_*^2 + \pi^2}{a_* \operatorname{sh} a_*} \left[\frac{C_* (a_*^2 + \pi^2)}{\operatorname{sh} a_*} \right]^{1/2} \tag{3.13}$$

In a metal ($P \ll 1$) the amplitude of the convection velocity field becomes

$$k = \pm 3.7 \varepsilon b_2 \sqrt{F b_1 P} \tag{3.14}$$

$$1 / b_1 = 4\pi a_* \varphi_{11}^2 (\operatorname{sh} a_* \operatorname{ch} a_* + \operatorname{sh} a_* - a_* - a_* \operatorname{ch} a_*)$$

$$1 / b_2 = 8\pi a_* \varphi_{21}^2 (\operatorname{sh} a_* \operatorname{ch} a_* + a_* \operatorname{ch} a_* - a_* - \operatorname{sh} a_*)$$

$$F = \frac{\operatorname{sh} 2a_* - 2a_*}{a_*} \left(\operatorname{ch} a_* + \frac{B}{a_*} \operatorname{sh} a_* + 1 \right) + \frac{C_* \operatorname{ch} a_*}{4} + \frac{C_* \operatorname{sh} a_* (1 - \operatorname{ch} 2a_*)}{8a_*^3}$$

Specifically, corresponding to these results, the amplitude of velocity in a slag with $B = 0$ is $k \approx \pm 50\varepsilon / P$ and in a liquid metal with $B = 1$ is $k \approx \pm 100\varepsilon\sqrt{P}$.

4. Convective Diffusion. For calculating the velocity at which gas escapes from a liquid layer, one must solve the equation of convective diffusion, which for the steady-state case in dimensionless variables is

$$P_D \left(v_\rho \frac{\partial c}{\partial \rho} + w \frac{\partial c}{\partial \zeta} \right) = \frac{\partial^2 c}{\partial \zeta^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial c}{\partial \rho} \quad \left(P_D = \frac{\nu}{D} \right). \tag{4.1}$$

Here P_D is the Prandtl diffusion number and c is the concentration of gas in the liquid.

The axial velocity component w is determined from Eqs. (2.4), (2.9), (2.11), and according to the continuity equation, the radial velocity component is

$$v_\rho = -(k / a_*) J_1(a_* \rho) dW / d\zeta . \tag{4.2}$$

Since usually for liquids $P_D \gg 1$, the condition $kP_D \gg 1$ $(k > 0)$ can be satisfied in a convection cell even at low amplitudes of velocity.

In this case one may apply the V. G. Levich theory of convective diffusion [4], according to which the concentration distribution of a substance in a volume is characterized by the presence of a thin diffusion layer resulting in a concentration change only within a thin layer near the surface $\zeta = 0$, where the velocity field is described by the first terms of a power series in ζ:

$$w = k\alpha_i J_0(a_* \rho)\zeta, \qquad v_\rho = -k(\alpha_i / a_*) J_1(a_* \rho) \tag{4.3}$$
$$(\alpha_1 = a_* - \operatorname{sh} a_* \operatorname{ch} a_*, \qquad \alpha_2 = a_*^2 - \operatorname{sh}^2 a_*) .$$

Subscripts $i = 1, 2$ correspond to convection in the slag and in the metal respectively.

The problem is now reduced to finding the solution to the equation

$$-k\alpha_i P_D \left[\frac{1}{a_*} J_1(a_* \rho) \frac{\partial c}{\partial \rho} - J_0(a_* \rho) \zeta \frac{\partial c}{\partial \zeta}\right] = \frac{\partial^2 c}{\partial \zeta^2} \tag{4.4}$$

with boundary conditions defining the concentration $c = c_\infty$ in the melt volume $(\zeta \to \infty)$ and in the vicinity of a confluence point (if $w < 0$ at $\rho = 0$) or line (if $w > 0$ at $\rho = 0$) and $c = c_0$ at the boundary surface $(\zeta = 0)$ except the confluence point or line.

The Mises transform [4] signifies a transition from variables ρ, ζ to new variables ψ, ρ, where

$$\psi = \frac{\rho\zeta}{a_*} J_1(a_* \rho) \quad \left(\frac{v_\rho}{k\alpha_i} = -\frac{1}{\rho}\frac{\partial \psi}{\partial \zeta}, \frac{w}{k\alpha_i} = \frac{1}{\rho}\frac{\partial \psi}{\partial \rho}\right). \tag{4.5}$$

Then Eq. (4.4) becomes the equation of heat conduction

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \psi^2} \quad \left(d\tau = -\frac{\rho^2}{k\alpha_i P_D a_*} J_1(a_* \rho) d\rho\right). \tag{4.6}$$

If $w > 0$ $(k < 0)$, then the confluence line of the current corresponds to $a_* \rho = x_{11}$, and therefore, function τ is conveniently chosen in the form

$$k\alpha_i P_D a_*^4 \tau_+ = [x_{11}^2 J_2(x_{11}) - a_*^2 \rho^2 J_2(a_* \rho)] \tag{4.7}$$

If $w < 0$ $(k > 0)$, then the confluence point obviously corresponds to $\rho = 0$, and therefore, one must choose

$$-k\alpha_i P_D a_*^2 \tau_- = \rho^2 J_2(a_* \rho) . \tag{4.8}$$

In both cases the boundary conditions are identical:

$$\begin{aligned} c &= c_0 \quad \text{for} \quad \psi = 0, \ \tau \neq 0 \\ c &\to c_\infty \quad \text{for} \quad \psi \to \infty \\ c &\to c_\infty \quad \text{for} \quad \tau \to 0, \ \psi \neq 0 . \end{aligned} \tag{4.9}$$

Equation (4.6) with boundary conditions (4.9) admits a self-simulating solution:

$$\frac{c - c_0}{c_\infty - c_0} = \operatorname{erf} \frac{\psi}{2\sqrt{\tau}} \quad \left(\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt\right). \tag{4.10}$$

In this way the diffusion current density at the interphase boundary is

$$j = -\frac{D}{h}\left(\frac{\partial c}{\partial \zeta}\right)_{\zeta=0} = -\frac{D(c_\infty - c_0)\rho J_1(a_*\rho)}{ha_*\sqrt{\pi\tau}}\,. \tag{4.11}$$

The total current from the surface area per one cell is

$$I = \frac{2\pi h^3}{a_*^2}\int_0^{x_{11}} jx dx = -\frac{2hx_{11}}{a_*^2}(c_\infty - c_0)\sqrt{\pi D\nu J_2(x_{11})\,|k\alpha_i|}\,. \tag{4.12}$$

The mean current density from a unit area of the melt surface is

$$I_0 = -\frac{D}{\delta}(c_\infty - c_0) \quad \left(\delta = \frac{x_{11}h\sqrt{\pi D}}{4\sqrt{\nu J_2(x_{11})\,|k\alpha_i|}}\right). \tag{4.13}$$

In specific cases the thickness of the diffusion boundary layer is

$$\delta \approx 0.1\,(D/\chi\varepsilon)^{1/2}, \quad P \gg 1, \quad B = 0$$
$$\delta \approx 0.1\,(D^2\chi/\nu^3\varepsilon^2)^{1/4}, \quad P \ll 1, \quad B = 1\,.$$

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